Math Logic: Model Theory & Computability
Lecture 22

Corollary (Compactness theorem). If a σ-theory T is finitely satisfiable, then it is satisfiable.
Proof. Since T is finitely satisfiable, every finite subtheory of it is consistent, but then the whole T is consistent, hence satisfiable by Gödel Completeness. $\square$

Henkin's proof of Gödel Completeness.

Given a consistent σ-theory T, the idea is to build an consistent an very $\tilde{\sigma}$-maximal extension $\tilde{T} \supseteq T$ so that $\tilde{T}$ contained information about how a potential model of $\tilde{T}$ should be defined.

Example. Given a σ-structure $\underline{A} := (A, \sigma)$, recall that $El\,Diag(\underline{A})$ is a theory in the signature $\sigma_A := \sigma \cup \{c_a : a \in A\}$, where c_a are constant symbols not present in σ. Furthermore:
$El\,Diag(\underline{A}) = \{\varphi(c_{a_1}, c_{a_2}, \ldots, c_{a_n}) : \underline{A} \models \varphi(a_1, a_2, \ldots, a_n),\ \varphi(x_1, \ldots, x_n) \text{ is an extended } \sigma\text{-formula}\}$.
Then clearly, having $El\,Diag(\underline{A})$, we can rebuild a model $\hat{\underline{A}}$ isomorphic to $\underline{A}$: indeed take $\hat{A} := \{c_a : c_a \in \tilde{\sigma} \setminus \sigma\}$ and define the interpretations of symbols in σ just like $El\,Diag(\underline{A})$ forces us to, e.g. put $f(c_{a_1}, c_{a_2}) := c_{a_3}$ iff $f(c_{a_1}, c_{a_2}) = c_{a_3} \in El\,Diag(\underline{A})$.
Note that not only $El\,Diag(\underline{A})$ is $\tilde{\sigma}$-maximal consistent, but it also has the additional property that whenever $\exists v \varphi \in El\,Diag(\underline{A})$ for some extended $\tilde{\sigma}$-formula $\varphi(v)$, then there is a constant symbol $c_a \in \tilde{\sigma}$ such that $\varphi(c_a/v) \in El\,Diag(\underline{A})$ just because $\underline{A} \models \exists v \varphi$ hence there could be a witness $a \in A$ to φ, i.e. $\underline{A} \models \varphi(a/v)$. Turns out that demanding this additional property, together with $\tilde{\sigma}$-maximal completeness, is enough to build a model even if the given theory is not of the form $El\,Diag(\underline{A})$.

Def. Let τ be a signature. A τ-theory H is called **Henkin** if it is τ-maximal consistent and for each extended τ-formula $\varphi(v)$, if $\exists v\varphi \in H$ then there is some $c \in Const(\tau)$ such that $\varphi(c/v) \in H$. We call this constant symbol c a **Henkin witness** for $\exists v \varphi$.

For a signature τ to be possible to admit a Henkin τ-theory, τ has to contain lots of constant symbols (at least one). So to build a Henkin theory extending a give consistent σ-theory T, we first need to extend the signature.

Adding Henkin witness to signature. Given a signature σ, we suppose for convenience (to avoid dealing with transfinite recursion/induction) that σ is ctbl. Then there are ctbly-many σ-formulas and we build a still ctbl extension σ_H of σ by $\sigma_H := \bigcup_{n\in\mathbb{N}} \sigma_n$, where $\sigma_0 := \sigma$ and each σ_n is still ctbl. We build the sequence $(\sigma_n)_{n\in\mathbb{N}}$ by induction on n. Set $\sigma_0 := \sigma$ and suppose that σ_n is defined. Put

$$\sigma_{n+1} := \sigma_n \cup \{c^{n+1}_{\exists v\varphi} : \varphi(v) \in ExtFormulas(\sigma_n)\}.$$

If σ_n is ctbl, then so is σ_{n+1}, which proves that σ_H is ctbl being a ctbl union of ctbl sets.

Lemma. Every consistent σ-theory T extends to a Henkin σ_H-theory.

Proof. Again we only prove for ctbl σ, since the idea of proof is the same in general. Let $(\sigma_n)_{n\in\mathbb{N}}$ be defined as above, with $\sigma_0 := \sigma$. Let T_0 be a σ_0-max consistent extension of T. We inductively build an increasing sequence $(T_k)_{k\in\mathbb{N}}$ such that each T_{2k} is σ_k-maximal consistent theory. Suppose T_{2k} is defined, define $T_{2k+1} := T_{2k} \cup \{\varphi(c^{k+1}_{\exists v\varphi}/v) : \exists v\varphi \in T_{2k}\}$. Part (c) of Lemma about consistency implies that if T_{2k} is consistent then T_{2k+1} is consistent.

Lastly, define T_{2k+2} as some σ_{k+1}-maximal consistent extension of T_{2k+1}. This finishes the inductive construction and we let $H := \bigcup_{k\in\mathbb{N}} T_k$, so H is a σ_H-theory. By the lemma about nested unions of consistent theories, H is consistent. Similarly, H is σ_H-maximal because for any σ_H-sentence φ, φ uses only finitely many constants, so φ is a σ_k-sentence for some $k\in\mathbb{N}$, hence $\varphi\in T_{2k}$ or $\neg\varphi\in T_{2k}$ because T_{2k} is σ_k-maximal. Also similarly, one verifies that H is Henkin: suppose $\exists v\varphi\in H$ for some extended σ_H-formula $\varphi(v)$. But then $\exists v\varphi\in T_{2k}$ for some $k\in\mathbb{N}$, hence $\varphi(c^{k+1}_{\exists v\varphi}/v)\in T_{2k+1}\subseteq H$. $\square$

To prove that a given consistent σ-theory T has a model, it is enough to take a Henkin extension $H\supseteq T$ to a σ_H-theory and build a model $\underline{M}_H := (M,\sigma_H)$ for H. Indeed, then the reduct $\underline{M} := (M,\sigma)$ of $\underline{M}_H$ to a σ-theory would be a σ-structure satisfying T. Thus to prove Gödel Completeness, it is enough to prove the following:

Main Lemma. Let τ be a signature. Every Henkin τ-theory H has a model.

To prove this, first note the following:

Lemma. Let H be a τ-Henkin theory and t be a τ-term. Then there is a (not necessarily unique) constant symbol $c\in\tau$ such that $t=c\in H$.

Proof. Because $t=t$ is an axiom for τ, we let $\varphi := t=v$, so $\varphi(t/v)$ is $t=t$, and $\varphi(t/v)\to\exists v\varphi$ is provable from the axioms of τ (this is the contrapositive of $(\forall v\neg\varphi)\to\neg\varphi(t/v)$, which is axiom (4)). Thus, $H\vdash t=t$ and $H\vdash\varphi(t/v)\to\exists v\varphi$, hence $H\vdash\exists v\varphi$ by MP, so $\exists v\varphi\in H$ by maximality. Because H is Henkin, there is $c\in\mathrm{Const}(\tau)$ with $\varphi(c/v)\in H$, i.e. $t=c\in H$. $\square$